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W_∞ Structures of 2D String Theory¹

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Abstract

The Ward identities of the W_∞ symmetry in 2D string theory in the tachyon background are studied in the continuum approach. Comparing the solutions with the matrix model results, it is verified that 2D string amplitudes are different from the matrix model amplitudes only by the external leg factors even in higher genus. This talk is based on the recent work [1] and also [2] for the $c_M < 1$ model.

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1 Introduction

Many interesting issues in string theory such as dynamical compactification, black hole physics, etc, require a non-perturbative formulation. At present, such a formulation is not available in higher dimensional string theories. In two or fewer spacetime dimensions, however, the string theory becomes solvable [3] due to the presence of W_∞ symmetry [4, 5, 6, 1, 2, 8, 7], which gives a possibility of studying the non-perturbative formulation of string theory. Furthermore 2D string theory itself has interesting spacetime physics. It gives the 2D dilaton gravity coupled to a massless field called “tachyon” as the effective theory [9]:

$$I_{\text{eff}} = \int d^2x \sqrt{-G} e^{-2\Phi} \left[\frac{1}{4} R^G + (\nabla\Phi)^2 + 4 - \frac{1}{2} (\nabla T)^2 + 2T^2 + \frac{2\sqrt{2}}{3} T^3 \right]. \quad (1)$$

Thus 2D string theory is also attractive as an alternative approach to studying 2D quantum dilaton gravity [10].

There are several formulations of two dimensional string theory. The matrix model (see reviews [3] and references therein) is generally believed to describe the 2D string theory, which is in principle defined non-perturbatively. The continuum theory [11, 12, 13] is defined using the standard quantization method of the string perturbation theory. The topological description of 2D string theory is formulated in refs.[15,7]. To understand the non-perturbative formulation of string theory it is important to clarify the relations between these methods.

In the present work we investigate the continuum method of 2D string theory and show that the following relation is exact even in higher genus:

$$S_{k_1, \dots, k_N \rightarrow p_1, \dots, p_M}^{(g)} = \prod_{i=1}^N \frac{\Gamma(-k_i)}{\Gamma(k_i)} \prod_{j=1}^M \frac{\Gamma(-p_j)}{\Gamma(p_j)} \hat{S}_{k_1, \dots, k_N \rightarrow p_1, \dots, p_M}^{(g)}, \quad (2)$$

where $S^{(g)}$ stands for 2D string amplitude of genus g and $\hat{S}^{(g)}$ is identified with the matrix model amplitude of genus g .

The strategy how to show that the above relation is indeed satisfied is the following. We can directly calculate the $1 \rightarrow N$ sphere amplitudes in the continuum method so that we can check the relation easily [3]. But, it is very difficult to calculate general sphere amplitudes and higher-genus ones in the continuum method. We here consider the Ward identities of the W_∞

symmetry which give the recursion relations between amplitudes on different genus. We then compare the solutions with the matrix model results.

2 Physical States and W_∞ Currents

We consider 2D string theory in the linear dilaton background [13], $G_{ij} = \delta_{ij}$ and $\Phi = 2\phi$, which is the vacuum solution of 2D dilaton gravity (1),

$$I_0 = \frac{1}{4\pi} \int d^2z \sqrt{\hat{g}} (\hat{g}^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + \hat{g}^{\alpha\beta} \partial_\alpha X \partial_\beta X + 2\phi \hat{R}) , \quad (3)$$

where ϕ is the Liouville field which is identified with the space coordinate and X is the $c_M = 1$ matter field which is identified with the (Euclidean) time.

A physical state with continuous momentum can only be the massless scalar called “tachyon” in the string terminology. In two dimensions the tachyon mode becomes massless in the linear dilaton vacuum. The tachyon vertex operator with momentum/energy $k(> 0)$ is given by

$$T_k^\pm = \frac{1}{\pi} \int d^2z e^{(2-k)\phi(z,\bar{z}) \pm ikX(z,\bar{z})} , \quad (4)$$

where \pm denotes the chirality. The selection of $k > 0$ is called the Seiberg condition [12]. We will postulate below that the amplitude including the anti-Seiberg ($k < 0$) states vanishes.

There is an infinite number of physical states at the integer momenta called the discrete states [16], which are constructed from the OPE of the tachyon operators with integer momenta, $V_{n+1}^-(z, \bar{z}) V_{m+1}^+(w, \bar{w}) \sim \frac{1}{|z-w|^2} R_{n,m}(w) \bar{R}_{n,m}(\bar{w})$, where n, m are positive integers² and $V_k^\pm(z, \bar{z})$ is the exponential part of the tachyon operator (4). The states $R_{n,m}$ are nothing but the remnants of the massive string modes in higher dimensional string theories. The discrete states $R_{n,m}$ form the chiral W_∞ algebra [4]. We here normalize the fields such that

$$R_{n,m}(z) R_{n',m'}(w) = \frac{1}{z-w} (nm' - n'm) R_{n+n'-1, m+m'-1}(w) . \quad (5)$$

²We slightly change the notation of the subscript of R and B in refs.[1,2].

Besides these, at the same momenta, there are the BRST invariant operators with conformal dimension zero, $B_{n,m}$ [16], which satisfy the ring structure $B_{n,m}(z)B_{n',m'}(w) = B_{n+n'-1, m+m'-1}(w)$. Combining $R_{n,m}(z)$ and $\bar{B}_{n,m}(\bar{z})$, we can construct the W_∞ symmetry currents [4] $W_{n,m}(z, \bar{z}) = R_{n,m}(z)\bar{B}_{n,m}(\bar{z})$, which satisfy

$$\partial_{\bar{z}}W_{n,m}(z, \bar{z}) = \{\bar{Q}_{BRST}, [\bar{b}_{-1}, W_{n,m}(z, \bar{z})]\} , \quad (6)$$

where the algebra $\partial_{\bar{z}} = \bar{L}_{-1} = \{\bar{Q}_{BRST}, \bar{b}_{-1}\}$ is used.

3 Scattering Amplitudes of Tachyons

Let us consider the action in the tachyon background

$$I = I_0 + \mu_B T_0 , \quad (7)$$

where $T_0 = \lim_{\epsilon \rightarrow 0} T_\epsilon^\pm$. The tachyon is massless so that S -matrix including T_0 vanishes. To ensure the non-decoupling of $\mu_B T_0$ we must make the bare tachyon background μ_B divergent as follows: $\mu_B \rightarrow \frac{\mu}{\epsilon}$.

The S -matrix of tachyons in the tachyon background is defined by

$$\begin{aligned} S_{k_1, \dots, k_N \rightarrow p_1, \dots, p_M}^{(g)} &= \langle \prod_{i=1}^N T_{k_i}^+ \prod_{j=1}^M T_{p_j}^- \rangle_g = \\ &\left(-\frac{\lambda}{2}\right)^{-\chi/2} \delta\left(\sum_{i=1}^N k_i - \sum_{j=1}^M p_j\right) \mu_B^s \frac{\Gamma(-s)}{2} \langle \prod_{i=1}^N T_{k_i}^+ \prod_{j=1}^M T_{p_j}^- (T_0)^s \rangle_g^{(free)} , \end{aligned} \quad (8)$$

The superscript *free* denotes the free field representation. The δ -function and $\mu_B^s \frac{\Gamma(-s)}{2}$ come from the zero-mode integrals of X and ϕ respectively [14]. g is the genus, $\chi = 2 - 2g$ and s is given by $s = \sum_{i=1}^N k_i + \chi - N - M$.

The theory is translationally invariant in the time X , while is not in the space coordinate ϕ . So the factorization property of amplitudes are different from the usual string theory in the zero-dilaton vacuum $\Phi = 0$. Let us introduce the eigenstate of the hamiltonian $H = L_0 + \bar{L}_0$ ³, $|h, l; N\rangle$ with the eigenvalue $\frac{1}{2}h^2 + \frac{1}{2}l^2 + 2N$, where $|h, l; N = 0\rangle = \bar{c}c \exp[(2 + ih)\phi(0, 0) + i\bar{l}X(0, 0)]|0\rangle$. The normalization is given by $\langle h', l'; N' | h, l; N \rangle_0 = -\frac{2}{\lambda}(2\pi)^2$

³ L_0 is the zero-mode of the Virasoro generator including the ghost part.

$\delta(h' + h)\delta(l' + l)\delta_{N', N}$. Note that the on-shell ($H = 0$) state has purely imaginary h . l must be real to preserve the translational invariance of X . The string propagator is given by $\frac{2}{H}$. So the factorization of 2D string amplitude into two parts is given in the form

$$\begin{aligned} \langle \mathcal{O} \rangle = & -\frac{\lambda}{2} \sum_{N=0}^{\infty} \int \frac{dh}{2\pi} \int \frac{dl}{2\pi} \langle \mathcal{O}_1 | -h, -l; N \rangle \\ & \times \frac{2}{\frac{1}{2}h^2 + \frac{1}{2}l^2 + 2N} \langle h, l; N | \mathcal{O}_2 \rangle + \dots, \quad (9) \end{aligned}$$

where \dots denotes other channels. The zero-mode integral of X ensures the conservation of energy so that l is fixed, while the zero-mode integral of ϕ does not produce the δ -function in the linear dilaton background. We then obtain the analytic function of h . So h integral is non-trivial even in tree amplitudes. Naively we can deform the h integral to the complex plane. It picks up the on-shell poles on the imaginary axis.

4 Ward Identities of W_{∞} Symmetry

We introduce the normalized tachyon vertex operator

$$\hat{T}_k^{\pm} = \Lambda(k) T_k^{\pm}, \quad \Lambda(k) = \frac{\Gamma(k)}{\Gamma(-k)} \quad (10)$$

and call the amplitude given by replacing T_k^{\pm} in (8) with \hat{T}_k^{\pm} the \hat{S} -matrix. Henceforth we consider the Ward identities in the form $\frac{1}{\pi} \int d^2z \partial_{\bar{z}} \langle W_{n,m}(z, \bar{z}) \mathcal{O} \rangle_g = 0$, where \mathcal{O} is a product of the normalized tachyon operators.

Let us first calculate the operator product expansion (OPE) between the current and the tachyon operators, which is given in refs.[4,1],

$$\begin{aligned} W_{n,m}(z, \bar{z}) \hat{T}_{k_1}^+(0, 0) \hat{T}_{k_2}^+ \cdots \hat{T}_{k_n}^+ \\ = \frac{1}{z} n! \left(\prod_{i=1}^n k_i \right) \hat{T}_{k_1 + \cdots + k_n - n + m}^+(0, 0), \quad (11) \end{aligned}$$

where $\hat{T}_k^+(z, \bar{z})$ is defined by replacing the integral in (4) with $\bar{c}(\bar{z})c(z)$. It was computed step by step from the $n = 1$ formula to the general n . This is analogous to the calculation of the OPE coefficients in CFT, where $\hat{T}_{k_j}^+$ ($j =$

$2, \dots, n$) just play a role of screening charges. Note that the OPE with the zero-momentum tachyon T_0 vanishes, but the OPE with the tachyon background $\mu_B T_0$ becomes finite due to the renormalization of μ_B .

The OPE with the tachyon \hat{T}_p^- is easily calculated by changing the chirality. It is carried out by changing the sign of the field X such that $\hat{T}^+ \rightarrow \hat{T}^-$ and $W_{n,m} \rightarrow -W_{m,n}$ (the roles of n and m are interchanged).

The OPE singularity gives the linear term of the Ward identity. In addition we get the BRST-trivial correlator $\langle \frac{1}{\pi} \int d^2 z \{ \bar{Q}_{BRST}, [\bar{b}_{-1}, W_{n,m}(z, \bar{z})] \} \mathcal{O} \rangle_g$. Usually such a correlator would vanish. In this case, however, it gives the anomalous contributions from the boundary of moduli space. The boundary is described by using the string propagator in the form

$$D = \frac{1}{\pi} \int_{e^{-\tau} \leq |z| \leq 1} \frac{d^2 z}{|z|^2} z^{L_0} \bar{z}^{\bar{L}_0} = \frac{2}{H} - \frac{2}{H} e^{-\tau H}, \quad \tau \rightarrow \infty, \quad (12)$$

where the second term of r.h.s. is the boundary.

Let us first calculate the $n = 1$ anomalous contribution. We then have to evaluate the following boundary contribution:

$$\begin{aligned} \lim_{\tau \rightarrow \infty} -\frac{\lambda}{2} \sum_{N=0}^{\infty} \int \frac{dl}{2\pi} \int \frac{dh}{2\pi} \int_{e^{-\tau} \leq |z| \leq 1} d^2 z < \mathcal{O}_1 [\bar{b}_{-1}, W_{1,m}(z, \bar{z})] \\ \times \bar{Q}_{BRST} \frac{-2}{H} e^{-\tau H} | -h, -l; N \rangle < h, l; N | \mathcal{O}_2 \rangle. \end{aligned} \quad (13)$$

We consider only the $N = 0$ mode. As a result, the $N \neq 0$ contributions vanish exponentially as $e^{-2N\tau}$. The z -dependence of the integrand is given in the form

$$\begin{aligned} [\bar{b}_{-1}, W_{1,m}(z, \bar{z})] \bar{Q}_{BRST} \frac{1}{H} e^{-\tau H} | -h, -l \rangle = \\ f(h, l) | z |^{\{(m-1)(-ih-l+2)-2m\}} e^{-\tau(h^2+l^2)/2} | -h + i(m-1), -l + m - 1 \rangle, \end{aligned} \quad (14)$$

where we use $\bar{Q}_{BRST} = \frac{1}{2} \bar{c}_0 H + \dots$. $f(h, l)$ is the calculable coefficient. Changing the variable to $|z| = e^{-\tau x}$, where $0 \leq x \leq 1$, we get the following τ -dependence: $2\pi\tau \exp[-\tau\{\frac{1}{2}(h^2+l^2) + x(m-1)(-ih-l)\}]$. Thus the integrand is highly peaked in the limit $\tau \rightarrow \infty$. So we can exactly evaluate the integral of h at the saddle point $h_{s.p.} = i(m-1)x$. We then get the expression

$$\begin{aligned} \lambda\tau \sqrt{\frac{2\pi}{\tau}} \int \frac{dl}{2\pi} \int_0^1 dx \exp\left[-\frac{\tau}{2}\{(m-1)x - l\}^2\right] f(h = i(m-1)x, l) \\ \times < \mathcal{O}_1 | i(m-1)(1-x), m-1-l \rangle < i(m-1)x, l | \mathcal{O}_2 \rangle. \end{aligned} \quad (15)$$

The x integral is also evaluated at the saddle point $x_{s.p.} = \frac{l}{m-1}$ and produces the coefficient $\frac{1}{m-1}f(h = il, l) = \Lambda(m-1-l)\Lambda(l)$. We then get the boundary contribution

$$\lambda \int_0^{m-1} dl < \mathcal{O}_1 \hat{T}_{m-1-l}^+ > < \hat{T}_l^+ \mathcal{O}_2 > , \quad (16)$$

where the Λ -factors are absorbed in the T_{m-1-l}^+ and T_l^+ . The l integral is restricted within the interval $0 \leq l \leq m-1$ because the saddle point $x_{s.p.}$ should be located within the interval $0 \leq x_{s.p.} \leq 1$ to give the finite contribution. Assuming that the boundary structure does not change in higher genus, we then get the expression

$$\lambda \int_0^{m-1} dl \left[\frac{1}{2} \sum_{\substack{\mathcal{O}=\mathcal{O}_1 \cup \mathcal{O}_2 \\ g=g_1+g_2}} < \hat{T}_{m-1-l}^+ \mathcal{O}_1 >_{g_1} < \hat{T}_l^+ \mathcal{O}_2 >_{g_2} + \frac{1}{2} < \hat{T}_{m-1-l}^+ \hat{T}_l^+ \mathcal{O} >_{g-1} \right] . \quad (17)$$

The second term is a variant of the first term, which comes from the configuration that two surfaces are connected by a handle. The factor $\frac{1}{2}$ in the first term is to correct the overcounting of the summation and that in the second term is to correct the double counting coming from the interchange of \hat{T}_{m-1-l}^+ and \hat{T}_l^+ .

We next consider the $n = 2$ case. We then have to evaluate the following quantity:

$$-\frac{\lambda}{2} \int \frac{dl}{2\pi} \int \frac{dh}{2\pi} < \mathcal{O}'_1 \left\{ \int_{e^{-\tau} \leq |z| \leq 1} d^2 z \bar{\partial} W_{2,m} \int_{|w| \leq |z|} d^2 w \hat{V}_k^+ + \int_{e^{-\tau} \leq |z| \leq 1} d^2 z \hat{V}_k^+ \int_{|w| \leq |z|} d^2 w \bar{\partial} W_{2,m} \right\} \frac{-2}{H} e^{-\tau H} | -h, -l > < h, l | \mathcal{O}'_2 > , \quad (18)$$

where $\hat{T}_k^+ = \int d^2 z \hat{V}_k^+(z, \bar{z})$. The primes on $\mathcal{O}_{1,2}$ denote the exclusion of the operator \hat{T}_k^+ . After carrying out the integration of w , we evaluate the z and h integral using the saddle point method. At $\tau \rightarrow \infty$ we obtain the contribution

$$\lambda 2! k \int_0^{m-2+k} dl < \mathcal{O}'_1 \hat{T}_{m-2+k-l}^+ >_{g_1} < \hat{T}_l^+ \mathcal{O}'_2 >_{g_2} , \quad (19)$$

where $g = g_1 + g_2$. There also is a variant of this contribution coming from the configuration where two surfaces are connected by a handle.

As an another variant of (19) we furthermore obtain the boundary contribution with the triple product of amplitudes,

$$\lambda 2! \int_0^{m-2} dl \int_0^{m-2-l} dl' < \hat{T}_{m-2-l-l'}^+ \mathcal{O}_1 >_{g_1} < \hat{T}_l^+ \mathcal{O}_2 >_{g_2} < \hat{T}_{l'}^+ \mathcal{O}_3 >_{g_3} , \quad (20)$$

where $g = g_1 + g_2 + g_3$. Noting the factorization property discussed before that the intermediate state becomes on-shell after integrating over the intermediate momentum, it is calculated by replacing the vertex operator \hat{V}_k^+ in (18) with the factorization formula $-\frac{\lambda}{2} \frac{2}{H_{l'}} \hat{V}_{-l'} < \hat{T}_{l'}^+ \mathcal{O}_2 >_{g_3}$, where $\frac{1}{H_{l'}} = \int \frac{dh}{2\pi} (\frac{1}{2}h^2 + \frac{1}{2}l'^2)^{-1} = \frac{1}{l'}$ and only the $N = 0$ mode is considered.

The general n formula is

$$\begin{aligned} & \lambda^{a-1} n! \left(\prod_{i=1}^{n+1-a} k_i \right) \int \prod_{i=1}^a dl_i \theta(l_i) \delta \left(\sum_{i=1}^a l_i - \sum_{i=1}^{n+1-a} k_i + n - m \right) \\ & \times < \hat{T}_{l_1}^+ \mathcal{O}'_1 >_{g_1} < \hat{T}_{l_2}^+ \mathcal{O}'_2 >_{g_2} \cdots < \hat{T}_{l_a}^+ \mathcal{O}'_a >_{g_a} , \quad (21) \end{aligned}$$

where $\sum_{i=1}^a g_i = g$ and $a = 1, \dots, n+1$. θ is the step function. The $a = 1$ formula is nothing but the contribution of the OPE (11). In addition, as discussed in the cases of $n = 1$, there are many variants of this expression coming from the boundary configurations that some of the surfaces are connected by handles.

The formulas with the vertex \hat{T}_p^- are given by changing the chirality; $\hat{T}^+ \rightarrow \hat{T}^-$ and $W_{n,m} \rightarrow -W_{m,n}$. Summarizing the boundary contributions, we can write out the Ward identities. For example, we get

$$\begin{aligned} 0 &= \frac{1}{\pi} \int \bar{\partial} < W_{2,1} \hat{T}_{k_1}^+ \hat{T}_{p_1}^- \hat{T}_{p_2}^- >_g \\ &= -x < \hat{T}_{k_1}^+ \hat{T}_{p_1}^- \hat{T}_{p_2}^- \hat{T}_1^- >_g - p_1 < \hat{T}_{k_1}^+ \hat{T}_{p_1+1}^- \hat{T}_{p_2}^- >_g \\ &\quad - p_2 < \hat{T}_{k_1}^+ \hat{T}_{p_1}^- \hat{T}_{p_2+1}^- >_g + 2! x k_1 < \hat{T}_{k_1-1}^+ \hat{T}_{p_1}^- \hat{T}_{p_2}^- >_g \\ &\quad - \frac{\lambda}{2} \int_0^1 dl < \hat{T}_{1-l}^- \hat{T}_l^- \hat{T}_{p_1}^- \hat{T}_{p_2}^- \hat{T}_{k_1}^+ >_{g-1} \\ &\quad + \frac{\lambda}{2} 2! k_1 \int_0^{k_1-1} dl < \hat{T}_{k_1-1-l}^+ \hat{T}_l^+ \hat{T}_{p_1}^- \hat{T}_{p_2}^- >_{g-1} \\ &\quad + \lambda 2! k_1 \sum_{h=0}^g \int_0^{k_1-1} dl < \hat{T}_{p_1}^- \hat{T}_{k_1-1-l}^+ >_h < \hat{T}_l^+ \hat{T}_{p_2}^- >_{g-h} , \quad (22) \end{aligned}$$

where the first term is given by the OPE with the tachyon background $\mu_B T_0$, where μ_B is replaced with the renormalized one $\mu = -x$. The second and the third terms are respectively given by the OPE with $\hat{T}_{p_1}^-$ and $\hat{T}_{p_2}^-$. The fourth term is given by the perturbed OPE with $\hat{T}_{k_1}^+$ and $\mu_B T_0$. The last three terms are just anomalous contributions coming from the boundary of moduli space.

This is not the end of the story. As for sphere amplitudes the W_∞ identities form a closed set among those and we can solve the identities recursively. We can obtain all types of sphere amplitudes of the normalized tachyons that exactly agree with the matrix model ones. For higher-genus cases, however, there is a problem. The $W_{n,m}$ identities with $n, m \leq 2$ such as (22) are consistent with the matrix model results. But, for the general $W_{n,m}$ identity with $n, m > 2$, further boundary contributions are necessary in order that the solution exists. It is easily imagined that there are the contributions shown in fig.1. On the basis of this figure we can speculate a generalisation of the formula (21) as follows:

$$\lambda^{a-1+h} \int \prod_{i=1}^a dl_i \theta(l_i) D_a^{+(h)}(-l_1, \dots, -l_a, k_1, \dots, k_{n+1-a-2h}; n, m) \\ \times < \hat{T}_{l_1}^+ \mathcal{O}'_1 >_{g_1} < \hat{T}_{l_2}^+ \mathcal{O}'_2 >_{g_2} \cdots < \hat{T}_{l_a}^+ \mathcal{O}'_a >_{g_a} , \quad (23)$$

where $\sum_{i=1}^a g_i = g - h$ and $a = 1, \dots, n+1-2h$. h stands for the genus of the surface Σ in fig.1. $-l_i$ represents the conjugate mode of $\hat{T}_{l_i}^+$. The Σ -part just gives the connectivity matrix $D_a^{+(h)}$ at $\tau \rightarrow \infty$. The $h = 0$ formula is nothing but (21). The $h \neq 0$ contributions exist for $g \geq h$ and $n \geq 2h$, where note that the $h = 1$ formula would contribute in the $W_{2,1}$ identities, but it vanishes due to the Seiberg condition.

The direct calculation of the connectivity matrix $D_a^{\pm(h)}$ for $h \geq 1$ is very difficult. So we guess the forms. Recall that the discrete state $R_{n,m}$ is given by the OPE of the tachyon operators $T_{n+1}^- \times T_{m+1}^+ \sim R_{n,m}$. This suggests that we could replace the operator $\bar{\partial}W_{n,m}$ with the two tachyons \hat{T}_{n+1}^- and \hat{T}_{m+1}^+ . Thus we identify the surface Σ with the $1 \rightarrow n+2-2h$ amplitude of genus h . From this argument we guess the expressions of D as follows:

$$D_a^{+(h)}(-l_1, \dots, -l_a, k_1, \dots, k_{n+1-a-2h}; n, m) \\ = \frac{\lambda^{1-h}}{(n+1)(m+1) \prod_{i=1}^a (-l_i)} \hat{S}_{m+1, -l_1, \dots, -l_a, k_1, \dots, k_{n+1-a-2h} \rightarrow n+1}^{(h)} , \quad (24)$$

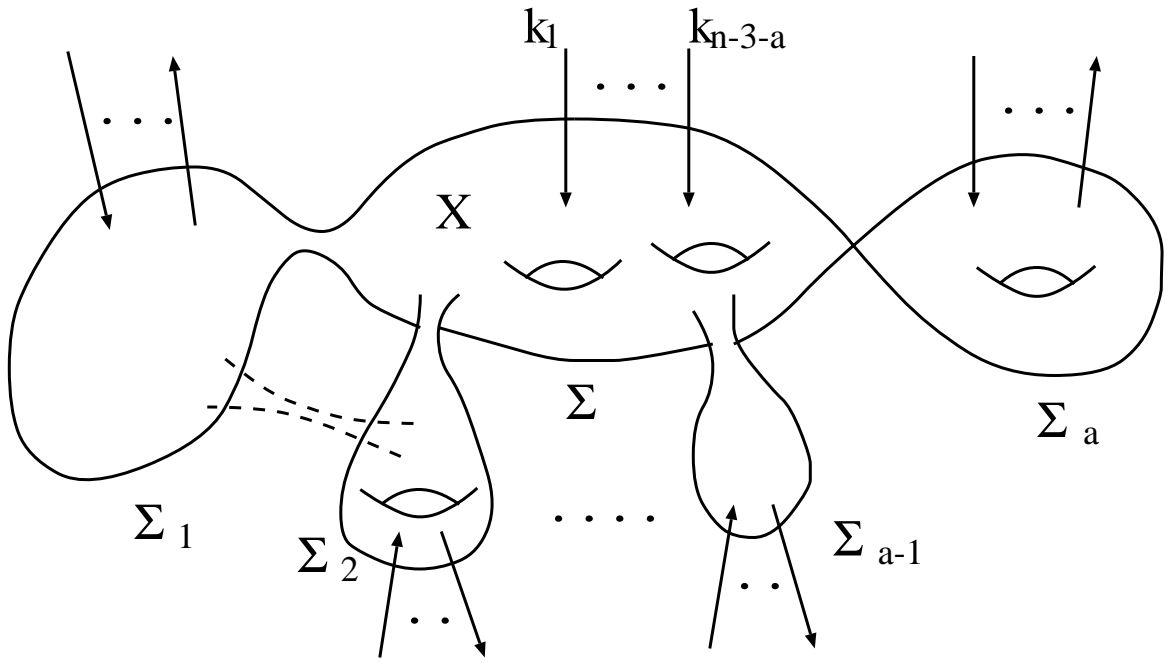


Figure 1: The incoming and the outgoing arrows denote \hat{T}_k^+ and \hat{T}_p^- , respectively. The cross point is $\bar{\partial}W_{n,m}$. The degenerate point of the surface stands for $\frac{-2}{H}e^{-\tau H}$.

where the \hat{S} -matrix formula is applied as if $-l_i$ were positive. The normalization is fixed by fitting the $h = 0$ formula with (21). For example, using the result of the matrix model [3], we obtain the genus-one expression

$$D_a^{+(1)} = \delta\left(\sum_{i=1}^a l_i - \sum_{i=1}^{n-1-a} k_i + n - m\right) \frac{1}{24} n! \left(\prod_{i=1}^{n-1-a} k_i\right) \\ \times \left(\sum_{i=1}^a l_i^2 + \sum_{i=1}^{n-1-a} k_i^2 + (m+1)^2 - n - 2\right). \quad (25)$$

Once the connectivity matrix of genus one $D_a^{\pm(1)}$ is given, we can then obtain all genus amplitudes recursively.

5 Summary

We studied the W_∞ structure of 2D string theory in the continuum method. We derived recursion relations which connect different genus amplitudes. For sphere amplitudes we can solve the W_∞ identities recursively and can obtain all types of amplitudes. For higher-genus cases we first checked that the $W_{n,m}$ identities with $n, m \leq 2$ are consistent with the matrix model. For general $W_{n,m}$, however, it is necessary to add the extra contributions (23) which are difficult to calculate directly. So we guessed the form from a simple argument and checked that the Ward identities are indeed closed and consistent with the matrix model. We explicitly verified the equivalence up to three genus by using the genus-one expression (25). In this way we conclude that the \hat{S} -matrix is equivalent with the matrix model amplitude in general genus.

Finally we comment on the work for the $c_M < 1$ model [2]. In this case we consider the chiral theory that consists of only the positive (or the negative) tachyon states with the discrete momenta. Then the W_∞ Ward identities result in the W -algebra constraints [17].

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References

- [1] K. Hamada, *Ward Identities of W_∞ Symmetry and Higher-genus Amplitudes in 2D String Theory*, KEK-TH-449, hep-th/9509025 (to be published in Nucl. Phys. **B**).
- [2] K. Hamada, Phys. Lett. **B324** (1994) 141; Nucl. Phys. **B413** (1994) 278.
- [3] I. Klebanov, *String Theory in Two Dimensions*, in “String Theory and Quantum Gravity”, Proceedings of the Trieste Spring School 1991, eds. J. Harvey et al. (World Scientific, Singapore, 1991); A. Jevicki, *Developments in 2D String Theory*, BROWN-HET-918, hep-th/9309115; P. Ginsparg and G. Moore, *Lectures on 2D Gravity and 2D String Theory*, YCTP-P23, LA-UR-923479; J. Polchinski, *What is String Theory*, NSF-ITP-94-97, hep-th/9411028.
- [4] E. Witten, Nucl. Phys. **B373** (1992) 187; I. Klebanov and A. Polyakov, Mod. Phys. Lett. **A6** (1991) 3273.
- [5] I. Klebanov, Mod. Phys. Lett. **A7** (1992) 723.
- [6] M. Fukuma, H. Kawai and R. Nakayama, Comm. Math. Phys. **143** (1992) 371; H. Itoyama and Y. Matsuo, Phys. Lett. **B262** (1991) 233.
- [7] J. Avan and A. Jevicki, Phys. Lett. **B266** (1991) 35; S. Das, A. Dhar, G. Mandal and S. Wadia, Int. J. Mod. Phys. **A7** (1992) 5165; Mod. Phys. Lett. **A7** (1992) 937; Erratum-ibid **A7**(1992) 2245; A. Dhar, G. Mandal and S. Wadia, Nucl. Phys. **B454** (1995) 541.
- [8] A. Hanany, Y. Oz and R. Plesser, Nucl. Phys. **B425** (1994) 150; D. Ghoshal and S. Mukhi, Nucl. Phys. **B425** (1994) 173; D. Ghoshal, C. Imbimbo and S. Mukhi, Nucl. Phys. **B440** (1995) 355; C. Imbimbo and S. Mukhi, Nucl. Phys. **B449** (1995) 553.
- [9] M. Natsuume and J. Polchinski, Nucl. Phys. **B424** (1994) 137.
- [10] K. Hamada, Phys. Lett. **B300** (1993) 322; K. Hamada and A. Tsuchiya, Int. J. Mod. Phys. **A8** (1993) 4897.
- [11] J. Distler and H. Kawai, Nucl. Phys. **B321** (1989) 509; F. David, Mod. Phys. Lett. **A3** (1988) 1651.
- [12] N. Seiberg, Prog. Theor. Phys. Suppl. **102** (1990) 319.
- [13] J. Polchinski, Nucl. Phys. **B346** (1990) 253.

- [14] M. Goulian and M. Li, Phys. Rev. Lett. **66** (1991) 2051; P. DiFrancesco and D. Kutasov, Nucl. Phys. **B375** (1992) 119.
- [15] S. Mukhi and C. Vafa, Nucl.Phys. **B407** (1993) 667.
- [16] P. Bouwknegt, J. McCarthy and K. Pilch, Comm. Math. Phys. **145** (1992) 541; B. Lian and G. Zuckermann, Phys. Lett. **B254** (1991) 417; Phys. Lett. **B266** (1991) 21.
- [17] M. Fukuma, H. Kawai and R. Nakayama, Int. J. Mod. Phys. **A6** (1991) 1385; R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. **B348** (1991) 435.